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Two-Dimensional Ising Model with Crossing and Four-Spin Interactions and a Magnetic Field $i(\pi/2)\,kT$

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The Ising model on a checkerboard lattice with crossing and four-spin interactions is solved exactly when there is pure imaginary magnetic field $H = i(\pi/2)\,kT$. The model exhibits a critical point with continuously varying exponents.

KEY WORDS: Ising model; pure imaginary field; second-neighbor interactions; exact solution.

1. INTRODUCTION

The two-dimensional Ising model in a nonzero magnetic field is a well-known unsolved problem in statistical physics. In 1952 Lee and Yang[(1)] obtained a solution for the two-dimensional nearest-neighbor model in the pure imaginary magnetic field

$$H = i\tfrac{1}{2}\pi kT \tag{1}$$

where T is the temperature. This solution, which has since been rederived from a variety of different approaches,[(2–6)] exhibits a second-order phase transition occuring at infinite temperature. This leads to the occurrence of a zero-temperature phase transition in a fully frustrated Ising model[(5,7)] in the dual space. It is also known that this solution of the nearest-neighbor model yields information on monomer correlations in the dimer problem.[2]

In this paper we show that the phase transition occuring in the two-dimensional Ising model at the pure imaginary field (1) behaves differently

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when there are crossing and/or multispin interactions. We consider, and exactly solve, an Ising model with nearest-neighbor, next-nearest-neighbor, and four-spin interactions on a checkerboard-type lattice and in the presence of the magnetic field (1).[8] Our analysis shows that the model exhibits a phase transition occuring at finite temperatures. Furthermore, the critical exponents are continuous varying, i.e., they are dependent on the interactions. It is a curious fact that this Ising model, while unsolvable in zero magnetic field, becomes solvable in the presence of the pure imaginary field (1).

2. DUALITY TRANSFORMATION

Consider an Ising model of N spins arranged on the square lattice as shown in Fig. 1. The four spins σ_1, σ_2, σ_3, and σ_4 surrounding each shaded square in Fig. 1 interact with an energy

$$\begin{aligned}E(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = &-J_1(\sigma_1\sigma_2+\sigma_3\sigma_4)-J_2(\sigma_2\sigma_3+\sigma_4\sigma_1)\\ &-J\sigma_1\sigma_3-J'\sigma_2\sigma_4-J_4\sigma_1\sigma_2\sigma_3\sigma_4\end{aligned} \tag{2}$$

as indicated in Fig. 2. In addition, there is an external magnetic field H which we shall set at the fixed value (1).

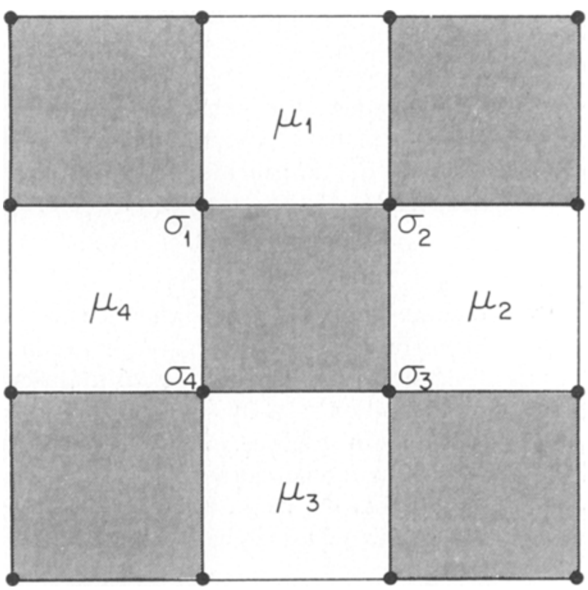

Fig. 1. The checkerboard Ising lattice.

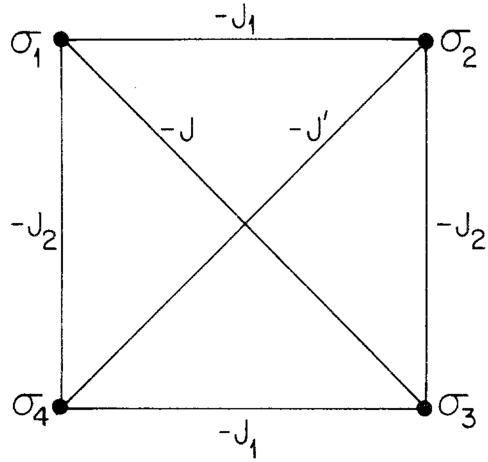

Fig. 2. Ising interactions (2) contained in a shaded square in Fig. 1. The four-spin interaction is not shown.

Since the thermodynamics of a system with complex Bolzmann factors may be boundary-condition-dependent, it is important to specify the precise boundary condition used. For our purposes we assume periodic boundary conditions. Write $L = H/kT$ and denote the partition function by $Z_N(L)$, where L in general can be complex. Then, by using the identity

$$e^{i\pi\sigma/2} = i\sigma \tag{3}$$

the partition function of the Ising model can be written, at $L = i\pi/2$, as

$$Z_N\left(i\frac{\pi}{2}\right) = i^N \sum_{\sigma_i = \pm 1} \prod_{\substack{\text{shaded} \\ \text{squares}}} B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{4}$$

where

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1\sigma_2 \exp[-E(\sigma_1, \sigma_2, \sigma_3, \sigma_4)/kT] \tag{5}$$

is the Boltzmann factor associated with a shaded square in Fig. 1. The factor i^N can be dropped if we assume N to be multiples of 4.[3]

Next we transform the partition function (4) into that of an Ising model with interactions in *every* square. This is a duality transformation wich can be effected in a number of different ways.[(9–11)] Here we follow a formulation due to Burkhardt,[(10)] which also permits a discussion of the spin correlation function, by placing the N/2 dual spins μ_l in the unshaded squares (cf. Fig. 1). It is then straightforward by following the procedure

[3] It can be quite easily verified that $Z_N(i\pi/2)$ is identically zero for $N = \text{odd}$.

given in Ref. 10 to rewrite the partition function (4) in the form of a spin summation in the dual space:

$$Z_N = \sum_{\mu_l = \pm 1} \prod_{\substack{\text{all}\\ \text{squares}}} W(\mu_1, \mu_2, \mu_3, \mu_4) \tag{6}$$

where

$$W(\mu_1, \mu_2, \mu_3, \mu_4) = \tfrac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} (-1)^{t_1\sigma_{12} + t_2\sigma_{23} + t_3\sigma_{34} + t_4\sigma_{41}} \times B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{7}$$

with

$$t_i = \tfrac{1}{2}(1 + \mu_i)$$

$$\sigma_{ij} = 1 - \delta_{kr}(\sigma_i, \sigma_j)$$

is the new "Boltzmann" weight for the dual Ising lattice. Note that this transformation is exact, and that the dual lattice has only $N/2$ spins and is oriented at a 45° rotation (cf. Fig. 1), also with periodic boundary conditions. It should also be noted that, when applied to the nearest-neighbor model, the duality transformation (7) corresponds to the decimation of half of the spins in the (fully frustrated) Ising model in the dual space, a procedure known to lead to an eight-vertex model at the decoupling point.[(12)]

For Boltzmann weights $B(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ such as those given by (5) satisfying the spin-reversal symmetry, the weights $W(\mu_1, \mu_2, \mu_3, \mu_4)$ are also spin-reversal invariant. We can then write (7) explicitly as

$$\mathbf{W} = \underline{X}\mathbf{B} \tag{8}$$

where **W** and **B** are column vectors whose components are[4]

$$\begin{aligned}
W_1 &= W(++++), & B_1 &= B(++++)\\
W_2 &= W(-+-+), & B_2 &= B(-+-+)\\
W_3 &= W(--++), & B_3 &= B(--++)\\
W_4 &= W(+--+), & B_4 &= B(+--+)\\
W_5 &= W(---+), & B_5 &= B(-+--)\\
W_6 &= W(-+--), & B_6 &= B(---+)\\
W_7 &= W(+---), & B_7 &= B(+---)\\
W_8 &= W(--+-), & B_8 &= B(--+-)
\end{aligned} \tag{9}$$

[4] Note the reversal roles of W_5, W_6 and B_5, B_6 with respect to spin arguments.

and $\underline{X}$ is the symmetric 8×8 matrix

$$\underline{X}=\frac{1}{2}\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{bmatrix} \tag{10}$$

Here $+(-)$ denotes $+1(-1)$. It can be easily verified that the inverse of (8) is

$$\mathbf{B}=2\underline{X}\mathbf{W} \tag{11}$$

3. EQUIVALENCE WITH AN EIGHT-VERTEX MODEL

For weights W which are invariant under spin reversals $\mu_l \to -\mu_l$, it is possible to introduce a (2–1) mapping of the spin configurations into the arrow configurations of an eight-vertex model.[13,14] This leads to the following exact equivalence:

$$Z_N = Z_{8v} \tag{12}$$

where Z_{8v} is the partition function of an eight-vertex model in the dual space whose vertex weights are

$$\omega_i = W_i, \qquad i=1, 2,..., 8 \tag{13}$$

Here, we have adopted the usual convention in numbering the vertices in effecting the mapping.[15,16]

It is now a simple matter to substitute (2) and (5) into (8), obtaining the following explicit expressions for the vertex weights:[5]

$$\{\omega_1, \omega_2,..., \omega_8\} = \{a_+, a_-, b_+, b_-, c_+, c_-, d_+, d_-\} \tag{14}$$

where

$$\begin{aligned} a=a_+=a_-&=(uvt)^{-1}(\sinh x + u^2v^2 \sinh y) \\ b=b_+=b_-&=(uvt)^{-1}(\sinh x - u^2v^2 \sinh y) \\ c_\pm &= (uvt)^{-1}[\cosh x + u^2v^2 \cosh y \mp t^2(u^2+v^2)] \\ d_\pm &= (uvt)^{-1}[\cosh x - u^2v^2 \cosh y \mp t^2(u^2-v^2)] \end{aligned} \tag{15}$$

[5] If we have started with a checkerboard lattice with two different horizontal (and vertical) nearest-neighbor interactions J_1, J_1' (and J_2, J_2'), then the resulting eight-vertex model has $a_+ \neq a_-$, $b_+ \neq b_-$, which has not been solved.

with

$$x = 2(K_1 + K_2), \qquad\qquad y = 2(K_1 - K_2)$$
$$u = e^{-K}, \qquad v = e^{-K'}, \qquad t = e^{-K4}$$
$$K_i = J_i/kT, \qquad K = J/kT, \qquad K' = J'/kT$$

Since the vertices with weights ω_5 and ω_6, and those with weights ω_7 and ω_8, occur in pairs in the eight-vertex model, we may replace both $c_\pm$ by c and $d_\pm$ by d where

$$c^2 = c_+ c_-, \qquad d^2 = d_+ d_- \tag{16}$$

It follows that the partition function (4) is precisely that of an eight-vertex model with standard weights a, b, c, d given by (15) and (16).

Baxter[17] has solved the eight-vertex model for real a, b, c, d. Thus, the partition function (4) can be evaluated in the regime $c_+ c_- > 0$, $d_+ d_- > 0$, or, equivalently,

$$|\cosh x \pm u^2 v^2 \sinh y| \geqslant t^2 |u^2 \pm v^2| \tag{17}$$

As is well known, the solution exhibits a transition with continously varying exponents, occuring at the critical point

$$|a| + |b| + |c| + |d| = 2\max\{|a|, |b|, |c|, |d|\} \tag{18}$$

4. FERROMAGNETIC MODEL

The above results are very general, applicable to ferromagnetic as well as antiferromagnetic interactions. For concreteness we now restrict ourselves to ferromagnetic interactions. It can be verified that for ferromagnetic interactions the vertex weights (16) are always positive and that the only possible realization of (18) is

$$|c| = a + b + |d| \tag{19}$$

which, after some reduction, reduces to

$$|\cosh y - t^4 \cosh x| = \sinh x[(u^{-4} - t^4)(v^{-4} - t^4)]^{1/2} \tag{20}$$

Here, without loss of generality, we have taken $K_1 \geqslant K_2 \geqslant 0$. Thus, the Ising model (2) is exactly solved at the fixed magnetic field (1) in the ferromagnetic regime. For pairwise interactions ($K_4 = 0$) for which the

Lee–Yang circle theorem[1] is valid, the critical condition (20) reduces further to

$$\coth 2K_1 + \coth 2K_2 = 2[(e^{4K} - 1)(e^{4K'} - 1)]^{-1/2} \tag{21}$$

We see that (21) yields a critical temperature T_c which is finite only when there are crossing interactions ($JJ' \neq 0$). The critical temperature diverges when there is no crossing interaction ($JJ' = 0$) and in the one-dimensional limit $J_1 = J_2 = 0$.

5. CORRELATION FUNCTION

We can apply the duality transformation (7) to the two-spin correlation function $\langle \sigma_{0,0}\sigma_{n,n} \rangle$, where $\sigma_{i,j}$ is the spin located at the point (i, j) in Fig. 1, to obtain an expression in the dual space. Writing

$$\sigma_{0,0}\sigma_{n,n} = (\sigma_{0,0}\sigma_{1,1})(\sigma_{1,1}\sigma_{2,2}) \cdots (\sigma_{n-1,n-1}\sigma_{n,n}) \tag{22}$$

and associating the factors $(\sigma_{\alpha,\alpha}\sigma_{\alpha+1,\alpha+1})$ to the appropriate shaded squares, we find

$$\langle \sigma_{0,0}\sigma_{n,n} \rangle = Z_{8v}^{(n)}/Z_{8v} \tag{23}$$

where $Z_{8v} = Z(i\pi/2)$ is the partition function of the eight-vertex model whose vertex weights are given by (15), and $Z_{8v}^{(n)}$ is the partition function of the same eight-vertex model with vertex weights along a *single* row of n sites modified to new values. These new weights are obtained from (7) with the replacement

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \to \sigma_1\sigma_3\, B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{24}$$

Inspection of (9) shows that this corresponds to negating B_3, B_4, B_7, and B_8 which, by virtue of (7), leads to the interchanges

$$W_1 \leftrightarrow W_4, \qquad W_2 \leftrightarrow W_3, \qquad W_5 \leftrightarrow W_8, \qquad W_6 \leftrightarrow W_7 \tag{25}$$

This further corresponds to the negation of the second-neighbor interactions in the spin representation (in the dual space) of the weights.[13,14]

Barber and Baxter[18] have obtained the magnetization of the eight-vertex model considered in the spin language of the dual space, and it is of interest to understand their result in the context of the Ising model under consideration. This is done by applying the inverse transformation (11) to the spin correlation function $\langle \mu_{0,0}\mu_{n,n} \rangle$. Again, writing

$$\mu_{0,0}\mu_{n,n} = (\mu_{0,0}\mu_{1,1}) \cdots (\mu_{n-1,n-1}\mu_{n,n}) \tag{26}$$

and associating the $(\mu_{\alpha,\alpha}\mu_{\alpha+1,\alpha+1})$ factors to the corresponding shaded squares in effecing the transformation, we obtain

$$\langle \mu_{0,0}\mu_{n,n}\rangle = Z_N^{(n)}\left(i\frac{\pi}{2}\right)\Big/ Z_N\left(i\frac{\pi}{2}\right) \tag{27}$$

where $Z_N(i\pi/2)$ is the partition function given by (4), and $Z_N^{(n)}(i\pi/2)$ is that of the same lattice but with signs of K_2, K, K' reversed in a row of $n/2$ adjacent shaded squares. This also corresponds to the interchange of the Boltzmann weights

$$B_1 \leftrightarrow B_3, \qquad B_2 \leftrightarrow B_4, \qquad B_5 \leftrightarrow B_7, \qquad B_6 \leftrightarrow B_8 \tag{28}$$

for these n shaded squares. Barber and Baxter's evaluation of the magnetization indicates that the expression (27) vanishes identically above T_c in the $n\to\infty$ limit.

6. LEE–YANG ZEROS

The Lee–Yang zeros are solutions of the equation

$$Z_N(L)=0 \tag{29}$$

in the complex $z=e^{-2L}$ plane, which, for ferromagnets, lie on the unit circle. In the limit of $N\to\infty$, the zeros attain a continuous distribution described by a density function $g(\theta)$, where θ is the azimuth angle of z. Lee and Yang[1] have shown that $4\pi g(\theta)$ is precisely the amount of discontinuity of the magnetization

$$I(L)=\frac{\partial}{\partial L}\lim_{N\to\infty}\frac{1}{N}\ln Z_N(L) \tag{30}$$

across the unit circle for θ fixed. Thus, the existence of a nonzero magnetization (and two-spin correlation function in the limit of an infinite separation) at $\theta=\pi$ necessarily implies $g(\pi)>0$, as is found to be the case in the one-dimensional and the two-dimensional nearest-neighbor models.[1,2] In both of these cases, we have $T_c=\infty$. The situation is less clear when there are second-neighbor interactions, since T_c is now finite. Certainly we must have $g(\pi)>0$ below T_c. For $T>T_c$ it is tempting to suppose that a gap will occur in the distribution of zeros across the negative axis, as is in the case at the positive real axis.[1] However, we know there is certainly one zero residing at $\theta=\pi$ for N odd. While we cannot rule out the possibility that the zeros may actually possess a vanishing density along an arc of the unit circle crossing the negative real axis, it appears more likely that the zeros are distributed continuously at all temperatures in the negative real half-plane. The function $g(\pi)$ will then exhibit some sort of singularity at T_c, perhaps vanishing above T_c. It would be useful to carry out numerical studies for large lattices to elucidate this point.

8. SUMMARY

We have obtained the exact solution of an Ising model with first-, second-, and four-spin interactions in the pure imaginary magnetic field $i\frac{1}{2}\pi kT$. The solution exhibits a phase transition only when there are nonzero crossing and/or four-spin interactions, and the transition possesses continuously varying exponents. We also obtained an expression in the dual space for the spin–spin correlation function and discussed possible forms of the Lee–Yang zero distribution across the negative axis.

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